Achieve Problem, Trig Substitution with some fixed $a > 0$.

$$I = \int x^2 \sqrt{a^2 - x^2}\, dx \overset{\text{Trig sub}}{=} \int a^2 \sin^2\theta \,(a\cos\theta)(a\cos\theta)\, d\theta$$

Trig Sub: $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$$
$$\sqrt{a^2-x^2} = a\cos\theta$$

$$= a^4 \int \sin^2\theta \cos^2\theta \, d\theta$$

$$= a^4 \int \left(\tfrac{1}{2}\right)(1-\cos(2\theta))\left(\tfrac{1}{2}\right)(1+\cos(2\theta))\, d\theta$$

$$= \frac{a^4}{2^2}\int 1 - \cos^2(2\theta)\, d\theta = \frac{a^4}{2^2}\int 1 - \left[\frac{1}{2} + \frac{1}{2}\cos(4\theta)\right] d\theta = \frac{a^4}{2^2}\int \frac{1}{2} - \frac{1}{2}\cos(4\theta)\, d\theta$$

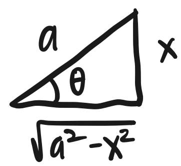
$$= \int \frac{a^4}{2^3} - \frac{a^4}{2^3}\cos(4\theta)\, d\theta = \frac{a^4}{2^3}\theta - \frac{a^4}{2^3}\left(\frac{1}{4}\right)\sin(4\theta) + C$$

$$= \frac{a^4}{2^3}\theta - \frac{a^4}{2^5}\left[2\sin(2\theta)\cos(2\theta)\right] + C$$

$$= \frac{a^4}{2^3}\theta - \frac{a^4}{2^4}(2\sin\theta\cos\theta)\left[1 - 2\sin^2\theta\right] + C$$

From Trig Sub:

$$\sin\theta = \frac{x}{a} \begin{matrix} : \text{opp} \\ : \text{hyp} \end{matrix}$$



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{a^2-x^2}}{a}$$

$$\theta = \arcsin\left(\frac{x}{a}\right)$$

$$= \frac{a^4}{2^3}\arcsin\left(\frac{x}{a}\right) - \frac{a^4}{2^3}\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2-x^2}}{a}\right)\left[1 - 2\left(\frac{x}{a}\right)^2\right] + C$$

$$\boxed{= \frac{a^4}{8}\arcsin\left(\frac{x}{a}\right) - \frac{a^2}{8}x\sqrt{a^2-x^2}\left(1 - \frac{2x^2}{a^2}\right) + C}$$

$$= \frac{a^4}{8}\arcsin\left(\frac{x}{a}\right) + \frac{1}{8}x\sqrt{a^2-x^2}\,(2x^2 - a) + C$$

Achieve Problem. Trig Substitution.

$$I = \int_0^{15} \sqrt{225 + x^2}\, dx \overset{\text{trig sub}}{=} \int_0^{\frac{\pi}{4}} 15\sec\theta\,(15\sec^2\theta)\, d\theta = 225 \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta \; ;$$

Trig Sub:
$$\begin{cases} x = 15\tan\theta, \quad dx = 15\sec^2\theta\, d\theta \\ \tan\theta = \frac{x}{15} \; ; \; \theta = \arctan\left(\frac{x}{15}\right) \\ \text{Bounds: } x = 15: \; \theta = \arctan\left(\frac{15}{15}\right) = \frac{\pi}{4} \\ \qquad\quad x = 0: \; \theta = \arctan(0) = 0 \\ 225 + x^2 = 15^2 + 15^2\tan^2\theta = 15^2\sec^2\theta \end{cases}$$

> Finding $I_2 = \int \sec^3\theta\, d\theta$:

$$I_2 = \int \sec^3\theta\, d\theta \overset{\text{IBP}}{=} \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \overset{\text{Pyth id.}}{=} \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

IBP: $u = \sec\theta$ $\quad dv = \sec^2\theta\, d\theta$
$\quad du = \sec\theta\tan\theta\, d\theta$ $\quad v = \tan\theta\, d\theta$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta = -I_2 + \sec\theta\tan\theta + \ln|\tan\theta + \sec\theta| \; ;$$

↑ Used table of integrals for this.

Sidenote: $\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right)dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$

u-sub $\{u = \sec(x) + \tan(x),\; du = \sec(x)\tan(x) + \sec^2(x)\,dx\}$

$= \ln|\sec(x) + \tan(x)| + C$

$$2I_2 = \sec\theta\tan\theta + \ln|\tan\theta + \sec\theta| \; ; \; I_2 = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\tan\theta + \sec\theta| + C \; ;$$

Set $F(\theta) = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\tan\theta + \sec\theta|$;

$$I = \int_0^{15} \sqrt{225 + x^2}\, dx = 225 \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = 225\left[F\left(\frac{\pi}{4}\right) - F(0)\right] \; ;$$

For $\theta = \frac{\pi}{4}$: $\tan\left(\frac{\pi}{4}\right) = 1$, $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\sec\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = 2^{1-\frac{1}{2}} = \sqrt{2}$;

$$F\left(\frac{\pi}{4}\right) = \frac{1}{2}(\sqrt{2})(1) + \frac{1}{2}\ln|1 + \sqrt{2}| = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1 + \sqrt{2}) \; ;$$

For $\theta = 0$: $\tan(0) = 0$, $\cos(0) = 1$, $\sec(0) = \frac{1}{1} = 1$;

$$F(0) = \frac{1}{2}(1)(0) + \frac{1}{2}\ln|1 + 0| = \frac{1}{2}\ln(1) = 0 \; ;$$

$$I = 225\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1 + \sqrt{2}) - 0\right] = \boxed{\frac{225}{2}\left(\sqrt{2} + \ln(1 + \sqrt{2})\right)} \; ;$$